Strong coupling effective Higgs potential and a first order thermal phase transition from AdS/CFT duality

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Abstract

We use AdS/CFT duality to study the thermodynamics of a strongly coupled $\mathcal{N} = 2$ super-symmetric large N_c $SU(N_c)$ gauge theory with $N_f = 2$ fundamental hypermultiplets. At finite temperature T and isospin chemical potential μ, a potential on the Higgs branch is generated, corresponding to a potential on the moduli space of instantons in the AdS description. For $\mu = 0$, there is a known first order phase transition around a critical temperature T_c. We find that the Higgs VEV is a suitable order parameter for this transition; for $T > T_c$, the theory is driven to a non-trivial point on the Higgs branch. For $\mu \neq 0$ and $T = 0$, the Higgs potential is unbounded from below, leading to an instability of the field theory due to Bose-Einstein condensation.

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1 Introduction

In its original form, AdS/CFT duality [1, 2, 3] relates theories of closed strings in asymptotically AdS spaces to large N_c gauge theories with matter in the adjoint representation. Fields in the fundamental representation may be added by including an open string sector through the introduction of branes probing the supergravity background. Much effort has gone into studying dualities of this type, motivated largely by the goal of finding a supergravity background dual to QCD.

The first example of AdS/CFT duality for a theory with fundamental representations related a conformal $\mathcal{N} = 2$ $Sp(N)$ gauge theory to string theory in $AdS_5 \times S^5/Z_2$, with D7-branes wrapping the Z_2 fixed surface with geometry $AdS_5 \times S^3$ [4, 5]. A different approach to flavour in AdS/CFT has been considered in [6]. In [7], the duality of [4, 5] was extended to an $\mathcal{N} = 2$ $SU(N_c)$ theory with N_f massive fundamental hypermultiplets, essentially by removing the $\mathbb{Z}_2$ orientifold, which was justified by the fact that a probe D7-brane wrapping a contractible S^3 does not lead to a tadpole requiring cancellation. Although the field theory is not asymptotically free, it has a UV fixed point in the strict $N_c \to \infty$ limit. Following this, there have been a number of papers generalizing the duality to confining theories with fundamental representations [8] -[19], including non-supersymmetric examples in which spontaneous chiral symmetry breaking by a $\bar{\psi}\psi$ quark condensate occurs [20] -[29].

At zero temperature and vanishing quark mass, the $\mathcal{N} = 2$ gauge theory has a non-trivial Higgs branch, i.e. a moduli space of vacua on which the scalar components of the fundamental hypermultiplets have expectation values. For finite quark mass, the moduli space includes a mixed Coulomb-Higgs branch: the fundamental hypermultiplets may have non-zero expectation values if the vector multiplet scalars have particular VEV's equal to the quark mass. For both the massless and massive case, the moduli space of field theory is described in the AdS picture by instanton configurations on the D7-branes [30, 31, 32, 33]. Self dual field strengths are solutions of the D7-brane equations of motion due to a conspiracy between the Yang-Mills and Wess-Zumino terms in the D7-brane action. In general, this conspiracy is destroyed by supersymmetry breaking deformations, which give rise to a potential on the moduli space of instantons, corresponding to an effective potential on the Higgs branch. We will compute this potential at finite isospin chemical potential and temperature. We focus on the potential generated on a slice of the moduli space corresponding to a single instanton centered at the origin. This slice is parameterized by the instanton size, which is dual to a particular Higgs VEV.

At finite isospin chemical potential, we find the expected result that the theory is destabilized via an effective negative mass squared for the moduli, leading to Bose-Einstein condensation. The negative mass squared term in the effective potential is related to the metric on the Higgs

branch, which is correctly reproduced by the dynamics of spinning instantons on the D7-branes. To stabilize the theory, a positive scalar mass which is larger than the chemical potential would need to be introduced.

Next we consider finite temperature deformations. Some finite temperature properties of theories with fundamental representations have been studied previously using AdS/CFT duality [20, 22, 34]. As shown in [20, 35], there is a finite temperature first order phase transition as the ratio of the temperature to the quark mass is varied in the $\mathcal{N} = 2$ theory of [7] (a similar transition was discussed in [22]). In the AdS description, this transition corresponds to a change in the topology of the D7-brane embedding in an AdS-Schwarzschild background.

We explore the finite temperature behavior of this theory in more detail by computing the effective potential generated on the Higgs component of the moduli space. We find that the Higgs expectation value is an order parameter for the first order phase transition. At $0 < T \leq T_c$ and $\mu = 0$, the Higgs VEV is driven to the origin of moduli space. However for $T > T_c$ we find a surprise; the instanton size is driven towards a non-zero value, suggesting the existence of a vacuum in which the theory is higgsed. This is in contrast to the weak coupling behavior of the theory, for which the one-loop finite temperature effective potential implies that the origin of moduli space is at least metastable.

The organisation of this paper is as follows. In section 2, we review the AdS description [30, 31] of the mixed Coulomb-Higgs branch in the Karch-Katz $\mathcal{N} = 2$ theory [7]. In section 3, we consider the same theory at finite chemical potential. Section 4 is devolved to the finite temperature case. In section 5 we briefly summarize our results and discuss future developments.

2 SUGRA dual of an $\mathcal{N} = 2$ theory with fundamental representations

We consider an $\mathcal{N} = 2$ gauge theory which is dual [7] to string theory in $AdS_5 \times S^5$ with N_f D7-branes wrapping a surface which is asymptotically $AdS_5 \times S^3$. The matter content of this gauge theory is that of the $\mathcal{N} = 4$ $SU(N_c)$ gauge theory together with N_f massive hypermultiplets in the fundamental representation. In $\mathcal{N} = 1$ superspace, the Lagrangian is

$$\mathcal{L} = \mathrm{Im} \left[\tau \int d^2\theta d^2\bar{\theta} \left(\mathrm{tr}\,(\bar{\Phi}_I e^V \Phi_I e^{-V}) + Q_i^\dagger e^V Q^i + \tilde{Q}_i^\dagger e^{-V} \tilde{Q}^i \right) \right.$$
$$\left. + \tau \int d^2\theta \left(\mathrm{tr}\,(\mathcal{W}^\alpha \mathcal{W}_\alpha) + W \right) + \tau \int d^2\bar{\theta} \left(\mathrm{tr}\,(\overline{\mathcal{W}}_{\dot{\alpha}} \overline{\mathcal{W}}^{\dot{\alpha}}) + \overline{W} \right) \right],$$

(1)

where the superpotential W is

$$W = \mathrm{tr}(\epsilon_{IJK}\Phi_I\Phi_J\Phi_K) + \tilde{Q}_i(m + \Phi_3)Q^i.$$

(2)

Coordinates									
0	1	2	3	4	5	6	7	8	9
	D3								
	D7								
	x^μ			y^m (or y, Ω_3)				Z^i	
					r				

Table 1: Index conventions

The superfields Q^i and $\tilde{Q}_i$, labeled by the flavor index $i = 1 \cdots N_f$, comprise the $\mathcal{N} = 2$ fundamental hypermultiplets.

This theory is not asymptotically free and, at finite N_c, the corresponding string background suffers from an uncancelled tadpole. However, as in [7], we focus strictly on the $N_c \to \infty$ limit with fixed N_f. In this case there is a non-trivial UV fixed point for the 't Hooft coupling, while the dual AdS string background does not suffer from a tadpole problem since the probe D7-branes wrap a contractible S^3.

In coordinates which will be convenient for our purposes, the $AdS_5 \times S^5$ background is

$$ds^2 = \frac{r^2}{R^2}dx^\mu dx^\mu + \frac{R^2}{r^2}(dy^2 + y^2 d\Omega_3^2 + \sum_{i=1}^{2} dZ^i dZ^i), \tag{3}$$

$$e^\Phi = g_s,$$

$$F_{(5)} = dC_{(4)} = 4R^4(V_{S^5} + {}^* V_{S^5}), \qquad C_{(4)}|_{0123} = \frac{r^4}{R^4}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,$$

$$R^4 = 4\pi g_s N_c \alpha'^2,$$

where V_{S^5} is the volume form on S^5, and $r^2 \equiv y^2 + Z^i Z^i$. The 't Hooft coupling in the dual gauge theory is $\lambda = g^2 N_c = g_s N_c$.

The fundamental hypermultiplets arise from N_f D7-branes embedded in this geometry. Their action is

$$S_{D7} = -T_7 \int d^4x d^4y \sqrt{-\det G_{ab}|_{PB}} = -T_7 \int d^4x \, dy \, d\Omega_3 \, y^3 \sqrt{1 + (Z'_i)^2}, \tag{4}$$

where G_{PB} is the pull-back of the metric, a prime indicates a derivative with respect to y, and we have assumed an embedding independent of the coordinates x as well as the coordinates on S^3. There is a manifest minimum of the action when $Z'_i = 0$. We will choose

$$Z^1 = m, \quad Z^2 = 0, \tag{5}$$

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other solutions being related by the $U(1)_R$ symmetry corresponding to rotations in the plane spanned by the Z^i. The induced metric is

$$ds^2 = \frac{r^2}{R^2}dx_{//}^2 + \frac{R^2}{r^2}(dy^2 + y^2 d\Omega_3^2),\tag{6}$$

with $r^2 = y^2 + m^2$. The parameter m corresponds to the mass of the fundamental hypermultiplets. For $m = 0$, the geometry is $AdS_5 \times S^3$, while for $m \neq 0$, the geometry approaches $AdS^5 \times S^3$ at large r. The S^3 component of the D7-geometry contracts to zero size at $r = m$. So long as N_f is held fixed in the limit $N_c \to \infty$ with fixed $\lambda = g_s N_c \gg 1$, one can neglect the back-reaction of the D7-branes on the bulk geometry.

2.1 The Higgs branch

When the theory has massless quarks, the fundamental scalars q^i and $\tilde{q}_i$ (denoting bottom components of chiral superfields by lowercase letters,) have non-zero expectation values on the Higgs branch[1] while the adjoint scalar ϕ_3 of the $\mathcal{N} = 2$ vector multiplet vanishes. For non-zero mass, m, and vanishing ϕ_3, the fundamental hypermultiplets are massive and a pure Higgs branch does not exist. However there is a mixed Coulomb-Higgs branch when ϕ_3 has an expectation value such that some components of the hypermultiplets are massless. An example of a point on a mixed Coulomb-Higgs branch is given by a diagonal ϕ_3 for which all but the last k entries are vanishing,

$$\phi_3 = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & -m & & \\ & & & & \ddots & \\ & & & & & -m \end{pmatrix}.\tag{7}$$

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[1]The scalars ϕ_1 and ϕ_2 belonging to the adjoint hypermultiplet may also have non-zero expectation values on the Higgs branch.

In this case, the F-flatness equations $\tilde{q}_i(\phi_3 + m) = (\phi_3 + m)q^i = 0$ permit fundamental hyper-multiplet expectation values in which only the last k entries of q^i and $\tilde{q}_i$ are non-zero,

$$
q^i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_1^i \\ \vdots \\ \alpha_k^i \end{pmatrix}, \qquad \tilde{q}_i = \begin{pmatrix} 0 & \cdots & 0 & \beta_{1i} & \cdots & \beta_{ki} \end{pmatrix}. \tag{8}
$$

There are additional F and D-flatness constraints which we have not explicitly written.

In string theory, nonzero entries in (8) physically correspond to D3-branes which are coincident with and dissolved within the D7-branes. Dissolved D3-branes can be viewed as instantons in the eight-dimensional world-volume theory on the D7-branes [36], due to the Wess-Zumino coupling

$$
S_{WZ} = \frac{T_7}{g_s}(2\pi\alpha')^2 \int C|_{PB}^{(4)} \wedge \mathrm{tr}(F \wedge F). \tag{9}
$$

There is a one to one map between the moduli space of Yang-Mills instantons and the Higgs branch of this $\mathcal{N} = 2$ theory. The ADHM constraints from which instantons are constructed [37] are equivalent to the F and D-flatness equations [38, 39] (see also [40] for a review).

2.2 Supergravity description of the Higgs branch

Because of the known one-one correspondence between instantons and the Higgs branch, one expects that instantons solve the equations of motion of the non-Abelian[2] Dirac-Born-Infeld action describing D7-branes embedded in (3) according to (5). The existence such solutions is a non-trivial consequence of AdS/CFT duality [30, 31, 41].

The effective action describing D7-branes in the AdS background (3) is

$$
S = -T_7(2\pi\alpha')^2 \left(-\frac{1}{g_s} \int d^8\xi \, C^{(4)} \wedge \mathrm{tr}(F \wedge F) + \int d^8\xi \, \frac{e^{-\Phi}}{4}\sqrt{-\det G}\, \mathrm{Tr}\left(F_{\alpha\beta}F^{\alpha\beta} \right) \right) + \cdots, \tag{10}
$$

where we have not written terms involving fermions and scalars. This action is the sum of a Wess-Zumino term, a Yang-Mills term, and an infinite number of corrections at higher orders in α' indicated by $\cdots$ in (10). The correspondence between instantons and the Higgs branch suggests that the equations of motion should be solved by field strengths which are self-dual with respect to a flat four-dimensional metric.

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[2]The existence of a Higgs branch requires at least two flavors, or two D7-branes.

In this paper, we work to leading order only in the large 't Hooft coupling expansion generated by AdS/CFT duality, which allows us to neglect the higher order terms in the α' expansion[3] of the action. Constraints on unknown higher order terms arising from the existence of instanton solutions, as well as the exactly known metric on the Higgs branch, were discussed in [30, 41].

The induced metric (6) can be written as

$$ds^2 = \frac{r^2}{R^2}dx^\mu dx^\mu + \frac{R^2}{r^2}\sum_{m=1}^{4} dy^m dy^m , \tag{11}$$

with $r^2 = y^m y^m + m^2$. Field strengths which are self dual with respect to the flat four-dimensional metric $ds^2 = \sum_{m=1}^{4} dy^m dy^m$ solve the equations of motion, due to a conspiracy between the Wess-Zumino and Yang-Mills term. Inserting the explicit AdS background values (3) for the metric and Ramond-Ramond four-form, into the action (10) for D7-branes embedded according to (5), with non-trivial field strengths only in the y^m directions, gives

$$\begin{aligned} S &= -T_7(2\pi\alpha')^2 \int d^4x\, d^4y\, \frac{r^4}{4g_s R^4}\left(-\frac{1}{2}\epsilon_{mnrs}F_{mn}F_{rs} + F_{mn}F_{mn}\right) \\ &= -T_7(4\pi\alpha')^2 \int d^4x\, d^4y\, \frac{r^4}{4g_s R^4}F_-^2 , \end{aligned} \tag{12}$$

where $F_{mn}^- = \frac{1}{2}(F_{mn} - \frac{1}{2}\epsilon_{mnrs}F_{rs})$. Field strengths $F_{mn}^- = 0$, which are self-dual with respect to the flat metric $dy^m dy^m$, manifestly solve the equations of motion[4]. These solutions correspond to points on the Higgs branch of the dual $\mathcal{N} = 2$ theory. If $m \neq 0$ this is, strictly speaking, a point on the mixed Coulomb-Higgs branch, with expectation values of the form (7), (8). We emphasize that in order to neglect the back-reaction due to dissolved D3-branes, we are considering a portion of the moduli space for which the instanton number k is fixed in the large N_c limit.

2.3   A slice of the Higgs branch

For simplicity, we consider the case $N_f = 2$, which is the minimum value for which a non-trivial Higgs branch exists. We will focus on a slice of the Higgs branch (or mixed Coulomb-Higgs branch) corresponding to a single instanton centered at the origin, $y^m = 0$.

In "singular gauge", the $SU(2)$ instanton (using the same conventions as [40]) is given by

$$A_\mu = 0, \qquad A_m = \frac{2Q^2 \bar{\sigma}_{nm} y_n}{y^2(y^2 + Q^2)}, \tag{13}$$

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[3]The dimensionless expansion parameter is $\alpha'/R^2 = 1/\sqrt{\lambda}$.
[4]Anti-instantons, with $F^+ = 0$, correspond to non-supersymmetric configurations which do not solve the equations of motion.

where Q is the instanton size, and

$$\bar{\sigma}_{mn} \equiv \frac{1}{4}(\bar{\sigma}_m\sigma_n - \bar{\sigma}_n\sigma_m), \qquad \sigma_{mn} \equiv \frac{1}{4}(\sigma_m\bar{\sigma}_n - \sigma_n\bar{\sigma}_m),$$

$$\sigma_m \equiv (i\vec{\tau}, 1_{2\times2}), \qquad \bar{\sigma}_m \equiv \sigma_m^\dagger = (-i\vec{\tau}, 1_{2\times2}). \tag{14}$$

with $\vec{\tau}$ being the three Pauli-matrices. We choose singular gauge, as opposed to the regular gauge in which $A_n = 2\sigma_{mn}y^m/(y^2 + Q^2)$, because of the improved asymptotic behavior at large y. In the AdS setting, the Higgs branch should correspond to a normalizable deformation of the AdS background at the origin of the moduli space. We see that the solution falls to zero at large y and the deformation in the interior is controlled by the dimension two parameter Q^2 which corresponds to the VEV of $\tilde{q}^i q_i$. The choice of singular gauge will be of particular use when computing the effects of the chemical potential. The singularity of (13) at $y^m = 0$ is not problematic for computations of physical (gauge invariant) quantities.

For $m = 0$ and zero instanton number, the background geometry preserves the symmetries $SO(2,4) \times SU(2)_L \times SU(2)_R \times U(1)_R \times SU(2)_f$. The $SO(2,4)$ isometry of the AdS^5 factor corresponds to the conformal symmetry of the dual gauge theory. The $SU(2)_L \times SU(2)_R \sim SO(4)$ acts in the obvious way on the coordinates y^m. The $SU(2)_L$ factor corresponds to a global symmetry, while $SU(2)_R$ and $U(1)_R$, which acts on (Z^1, Z^2), correspond to R-symmetries[5]. $SU(2)_f$ is a gauge transformation on the D7-branes which is constant at the boundary of AdS. The action at the boundary of AdS corresponds to the flavor symmetry of the dual gauge theory.

In the presence of the instanton (13), and for $m \neq 0$, the symmetries are broken to $SO(1,3) \times SU(2)_L \times \text{diag}(SU(2)_R \times SU(2)_f)$ corresponding to a point on the mixed Coulomb-Higgs branch

$$q_{i\alpha} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \epsilon_{i\alpha}Q \end{pmatrix}, \quad \phi_\alpha = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \\ & & & -m \end{pmatrix}, \tag{15}$$

where $q_{i\alpha}$ are the scalar components of the fundamental hypermultiplets, with flavor index $i = 1, 2$ and $SU(2)_R$ index $\alpha = 1, 2$. The scalars ϕ_α belong to the adjoint hypermultiplet, while ϕ_3 is the adjoint scalar belonging to the vector-multiplet.

In general, one expects that supersymmetry breaking deformations, such as a finite chemical potential or temperature, lift the vacuum degeneracy of the Higgs and Coulomb branch. In the dual supergravity description, the lifting of the Higgs branch at finite temperature occurs because the conspiracy between the Wess-Zumino and Yang-Mills terms, necessary for self-dual field strengths to be solutions, no longer occurs. The potential which is generated on the Higgs branch can be computed by evaluating the D7-brane action on the space of self-dual field

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[5]There is no apparent distinction between $SU(2)_L$ and $SU(2)_R$ in the geometry of the AdS background. The distinction arises from the gauge field couplings to the RR four-form.

strengths. In the following we will compute the potential $V(Q)$ generated on the slice of the Higgs branch dual to the single instanton (13).

3   Finite chemical potential, zero temperature

As the simplest example of a potential generated on the Higgs branch, we first consider the case of finite chemical potential and zero temperature. For two flavors, there is a U(2) flavor symmetry, with Cartan generators 1 and σ_3, corresponding to baryon number and isospin respectively. We will consider a nonzero chemical potential for the isospin[6]. To include the chemical potential [42] we allow a spurious gauge field associated with the τ^3 component of isospin to acquire a VEV, μ, in its A^0 component. This includes generic fermion and scalar Lagrangian terms for fields with isospin charge e of the form

$$\delta\mathcal{L} = -\mu e\bar{\psi}\tau^3\gamma^0\psi + \mu^2 e^2|\phi|^2 \, . \tag{16}$$

The first term is a source for the fermionic isospin number density. In the path integral, this term places the theory at finite density. The second term is an unbounded scalar potential which renders the theory unstable, such that Bose-Einstein condensation is expected. We will reproduce this run-away behaviour of the scalar potential using the AdS/CFT description. Equivalently, we may set $A_\tau = 0$ by a gauge transformation $e^{i\mu e\tau}$, and perform the path integral with boundary conditions corresponding to a spinning configuration space.

AdS/CFT duality relates global symmetries of the boundary theory to gauge symmetries in the bulk supergravity. The prescription for turning on a chemical potential in the AdS description is to turn on a background (non-normalizable) flat gauge connection $\mathcal{A}_\tau = \mu$ for the associated gauge symmetry. Equivalently, one may consider a spinning AdS background. Examples of the AdS description of a chemical potential in various contexts have appeared in [43]-[48].

The effect of the chemical potential on the Higgs branch can be studied in the AdS description by computing the action for rigidly rotating instantons with moduli $\mathcal{M}^i$,

$$A_n = e^{i\mu t\sigma_3}\mathcal{A}_n^{\text{instanton}}(y^m, \mathcal{M}^i)e^{-i\mu t\sigma_3} \, . \tag{17}$$

Equivalently, we may add a background A^0 in the fixed instanton background,

$$A^0 = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \qquad A_n = A_n^{\text{instanton}} \, . \tag{18}$$

On the slice of the Higgs branch corresponding to the single instanton configurations (13) with modulus Q, the effective potential at quadratic order in μ can be determined by inserting (18)

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[6]Turning on a chemical potential for the baryon number has no apparent effect here since none of the fields on the D7 brane world volume are charged under this symmetry. This was pointed out to us by A. Karch.

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into the D7-brane action $\int d^4x\, V(Q) = -S_{D7}$ giving

$$V(Q) = T_7 \frac{(2\pi\alpha')^2}{g_s} \int d^4y \, \mathrm{tr} \left(\frac{1}{2} \frac{(y^2 + m^2)^2}{R^4} F^-_{mn} F^-_{mn} + 2 F_{m\mu} F_{m\nu} \eta^{\mu\nu} \right. \tag{19}$$
$$\left. + \frac{R^4}{(y^2+m^2)^2} F_{\mu\nu} F_{\alpha\beta} \eta^{\mu\alpha} \eta^{\nu\beta} \right) ,$$

with $y^2 = y^m y^m$. We have split the action into the pieces involving F in the x and y directions, indicated by Greek and Roman indices respectively, as well as mixed terms. For the background (18), the only non-zero contribution to the potential comes from the mixed term $\mathrm{tr}\, F_{\mu m} F_{\nu m} \eta^{\mu\nu} = -\mathrm{tr}\, [A_0, A_n]^2$, giving

$$V(Q) = -T_7 \frac{2(4\pi\alpha')^2}{g_s} \mu^2 \int d^4y \frac{Q^4}{y^2(y^2+Q^2)^2} = -T_7 \frac{2(4\pi^2\alpha')^2}{g_s} \mu^2 Q^2 . \tag{20}$$

Note that the term $\sqrt{g} F_{\mu m} F^{\mu m}$ in the D7-action is also the term which determines the metric (two-derivative effective action) on the Higgs branch. To quadratic order in the chemical potential, the result (20) is exact. Higher dimension operators in the DBI action with two Greek indices, which have been omitted from (19), must vanish in the instanton background, otherwise there would be corrections to the Higgs branch metric in a strong 't Hooft coupling expansion [30, 31]. By the same token, the term in the potential on the Higgs branch which is quadratic in the chemical potential is given exactly by the $F_{\mu m} F^{\mu m}$ term.

In light of (20), the $\mathcal{N} = 2$ theory is unstable at finite chemical potential. Physically, the instability is due to Bose-Einstein condensation. This instability will not be cured by taking into account the higher derivative terms in the effective action, whose contribution does not effect the large Q asymptotics. Potentially, the instability could be removed by the inclusion of an appropriately large mass term for the scalars, either as an explicit additional perturbation, or generated radiatively from the inclusion of some other relevant perturbation. The finite temperature configurations studied below, however, will not be able to cure this instability since - as shown below - the relevant deformation, T^4, has dimension four and can at best generate a potential T^8/Q^4. This does not stabilize the theory.

4 Finite temperature, zero chemical potential

Before investigating the large 't Hooft coupling behavior of the effective potential on the Higgs branch, let us review the situation to one loop in perturbation theory. In a supersymmetric theory, the one-loop effective potential is

$$V = V_0 + \frac{1}{8} T^2 \sum_{i,j} \left| \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \right|^2 , \tag{21}$$

where V_0 is the zero temperature effective potential, W is the superpotential, and ϕ_i are the scalar components of chiral superfields. For the theory we are considering, this potential drives the system towards the origin of moduli space. In fact it is often said that supersymmetric theories with flat directions do not have symmetry breaking vacua at finite temperature [49] (for some exceptions, see [50, 51]). However, for large 't Hooft coupling, we will find that the theory we consider does indeed exhibit symmetry breaking at high temperature. The origin of the Higgs branch becomes a maximum of the effective potential for $T > T_c$.

The gravitational dual of $\mathcal{N} = 4$ gauge theory at large 't Hooft coupling and finite temperature is obtained by replacing the background (3) with the AdS-Schwarzschild black-hole background [53]. The latter belongs to a general class of supergravity solutions which, in a choice of coordinates convenient for our purposes, have the form

$$ds^2 = f(r)(d\vec{x}^2 + g(r)d\tau^2) + h(r)(\sum_{m=1}^{4} dy^m dy^m + \sum_{i=1}^{2} dZ^i dZ^i),$$

$$e^{-\Phi} = \phi(r),$$

$$F^{(5)} = 4R^4(V_{S^5} + {}^*V_{S^5}) = dC_{(4)}, \qquad C_{(4)}|_{0123} = s(r)\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,$$

$$r^2 = y^m y^m + Z^i Z^i. \tag{22}$$

For the AdS-Schwarzschild solution, we have

$$f(r) = \frac{4r^4 + b^4}{4r^2 R^2}, \quad g(r) = \left(\frac{4r^4 - b^4}{4r^4 + b^4}\right)^2, \quad h(r) = \frac{R^2}{r^2}, \quad s(r) = \frac{r^4}{R^4}\left(1 + \frac{b^8}{16r^8}\right). \tag{23}$$

The coordinates $\vec{x}$ are the spatial coordinates of the dual gauge theory and τ is the Euclidean time direction, which is compactified on a circle of radius b^{-1}, corresponding to the inverse temperature. Note that the temperature $T \sim b$ only enters to the fourth power.

The D7 embedding in this background to describe our $\mathcal{N} = 2$ gauge theory with quarks is

$$Z^2 = 0, \qquad Z^1 = z(y), \tag{24}$$

where $y^2 \equiv y^m y^m$. The induced metric $G|_{\rm pb}$ takes the form

$$ds^2_{\rm D7} = f(r)\left(d\vec{x}^2 + g(r)d\tau^2\right) + h(r)\left(dy^m dy^m + \left(\frac{z'(y)}{y}\right)^2 (y^m dy^m)^2\right), \tag{25}$$

with $r^2 = y^2 + z^2(y)$. To lowest order in α', the D7-brane action is

$$-T_7 \int d^8\xi\, e^{-\Phi}\sqrt{-\det G}|_{\rm pb} = -2\pi^2 T_7 \int d^4x\, dy\, y^3 \phi(r) f(r)^2 \sqrt{g(r)}\, h(r)^2 \sqrt{1 + z'(y)^2}, \tag{26}$$

and the specific embeddings $z(y)$ can be found by solving the Euler-Lagrange equation with boundary conditions $z(\infty) = m$, $z'(\infty) = 0$.

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At large y the geometry returns to AdS where the asymptotic embedding solution (24) takes the form $z(y) \sim m + c/y^2$. Given m, the parameter c, corresponding to a $\bar{\psi}\psi$ condensate, is determined by requiring the solution to be smooth and normalizable. These solutions were studied in detail in [20].

The potential generated on the Higgs branch is obtained by calculating a contribution to the D7-brane action involving the gauge field F_{ab} at second order in α'. Specifically, one evaluates the action on the space of field strengths which are self-dual[7] with respect to the induced metric in the directions transverse to $\tau, \vec{x}$;

$$V = \frac{T_7(2\pi\alpha')^2}{2} \left(\frac{1}{g_s} \int d^4y \, C^{(4)}_{0123} \, \epsilon_{mnrs} \, \mathrm{Tr} \, F_{mn}F_{rs} - \frac{1}{2} \int d^4y \, e^{-\Phi} \sqrt{-\det G} \, \mathrm{Tr} \, F^{mn}F_{mn} \right), \quad (27)$$

where F_{mn} is self-dual with respect to the metric

$$ds_\perp^2 = h(r) \left((1 + z'(y)^2)dy^2 + y^2 d\Omega_3^2 \right). \quad (28)$$

This metric is conformally flat. With new coordinates $\tilde{y}(y)$ such that $ds^2 = \alpha(\tilde{y})(d\tilde{y}^2 + \tilde{y}^2 d\Omega_3^2)$, the instanton configurations (self-dual field strengths) take the usual form.

In the $b = 0$ (zero temperature) limit, the analysis of section (2.2) holds and the potential (27) becomes flat. At finite temperature, the conspiracy between the Wess-Zumino term and the Yang-Mills terms fails, giving a potential on the moduli space of instantons. We now compute the form of the potential for the slice of moduli space corresponding to an instanton centered at the origin.

4.1 Computation of the Higgs Effective Potential

We wish to evaluate the D7-brane action on the space of fields strengths which are self-dual with respect to the transverse part of the metric

$$ds_\perp^2 = h(r) \left(dy^m dy^m + \left(\frac{z'(y)}{y} \right)^2 (y^m dy^m)^2 \right).$$

This metric is conformally flat. There exists a coordinate transformation

$$\tilde{y}^m = J(y)y^m \quad (29)$$

such that the induced metric has the form

$$ds_{D7}^2 = f(r) \left(d\vec{x}^2 + g(r)d\tau^2 \right) + \frac{h(r)}{J(y)^2} d\tilde{y}^m d\tilde{y}^m. \quad (30)$$

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[7]There are couplings between world-volume scalars and field strengths at higher orders in α' which could alter the embedding. However we only consider the leading term in a large 't Hooft coupling expansion for which these couplings can be neglected.

The function $J(y)$ is obtained by solving

$$\left(\frac{J'}{J}\right)^2 + \frac{2}{y}\frac{J'}{J} - \frac{1}{y^2}z'(y)^2 = 0\,, \tag{31}$$

subject to the boundary condition $J(y \to \infty) = 1$. In the coordinates $\tilde{y}^m$, the single instanton centered at the origin has the usual form,

$$A_m = \frac{2Q^2\bar{\sigma}_{nm}\tilde{y}^n}{\tilde{y}^2(\tilde{y}^2 + Q^2)}\,. \tag{32}$$

The embedding function $z(y)$ is determined by shooting techniques as in [20]. The potential on the Higgs branch is then given by $S_{D7} = -\int d^4x V(Q)$, or

$$V(Q) = T_7\frac{(2\pi\alpha')^2}{2}\left(\frac{1}{g_s}\int d^4y\, C^{(4)}_{0123}\,\epsilon_{mnrs}\,\mathrm{Tr}\,F_{mn}F_{rs} - \frac{1}{2}\int d^4y\, e^{-\Phi}\sqrt{-\det G}\,\mathrm{Tr}F^{mn}F_{mn}\right)$$

$$= T_7\frac{(2\pi\alpha')^2}{4g_s}\int d^4\tilde{y}\left(s(r) - \phi(r)f(r)^2\sqrt{g(r)}\right)\frac{96\,Q^4}{(\tilde{y}^2 + Q^2)^4} \tag{33}$$

$$= T_7\frac{(2\pi\alpha')^2}{4g_s}Vol(S_3)\int y^3 dy\left(s(r) - \phi(r)f(r)^2\sqrt{g(r)}\right)\frac{96\,Q^4\,J(y)^4}{(J(y)^2y^2 + Q^2)^4}\left(1 + y\frac{J'(y)}{J(y)}\right)\,.$$

For the AdS-Schwarzschild background (22,23) we obtain

$$V(Q) = T_7\frac{6(2\pi^2\alpha')^2}{g_s R^4}\int dy\,\frac{b^8 y^3}{(J(y)^2y^2 + z(y)^2)^2}\frac{Q^4 J(y)^4}{(J(y)^2y^2 + Q^2)^4}\left(1 + y\frac{J'(y)}{J(y)}\right)\,. \tag{34}$$

4.2 Large VEV Asymptotics

Without resorting to numerics, some qualitative features of $V(Q)$ at large Q (or large instanton size) can be determined from the large y behavior of the induced metric and RR-form given in (22,23), together with the asymptotic behaviour of the embedding,

$$z(y) = m + \frac{c}{y^2} + \cdots\,. \tag{35}$$

At large y we can neglect the condensate and the leading terms in the potential will depend on m and on the deformation parameter of the background b. The D7-brane action for a large instanton, which has support at large y is

$$\int d^4x V(Q) = -S_{D7}$$

$$\approx T_7\frac{6(2\pi^2\alpha')^2}{g_s R^4}\int d^4x\int dy\, y^3\frac{Q^4}{(y^2 + Q^2)^4}\left[\frac{b^8}{y^4} - \frac{2b^8 m^2}{y^6} + \cdots\right] \tag{36}$$

$$= T_7\frac{6(2\pi^2\alpha')^2}{g_s R^4}\left[-\frac{b^8}{Q^4}\left(\frac{1}{2}\log 2 - \frac{1}{3}\right) + \frac{2m^2 b^8}{Q^6}\left(\frac{67}{48} - 2\log 2\right) + \cdots\right]\,.$$

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This asymptotic approximation is valid when $y \geq Q$ but gives IR divergent integrals. We have evaluated the integrals down to $y/Q = 1$. In the next section we evaluate the full potential including the correct, convergent, IR behaviour.

The asymptotic potential terms are simply what one would expect on dimensional grounds. The potential must vanish as $b \to 0$ and the geometry becomes AdS. Since b enters as b^4, the first Q dependent term must take the form b^8/Q^4.

4.3   Numerical computation of $V(Q)$

To compute $V(Q)$ in general requires knowledge of the embedding function $z(y)$, which has been computed by a numerical shooting technique in [20]. Imposing boundary conditions corresponding to the large y behaviour (35), and requiring smooth behaviour in the interior, such that an RG flow interpretation is possible, leads to a dependence $c(m, b)$ of the chiral quark condensate $\langle \bar{\psi}\psi \rangle$ on the quark mass and on the temperature. Depending on the ratio m/b, there are two types of solutions, which differ by the topology of the D7-branes. At large r (or y) the geometry of the D7-branes is $AdS_5 \times S^3$ and the topology of the $r \to \infty$ boundary is $S^1 \times R^3 \times S^3$. For sufficiently large m/b, the S^3 component of the D7-geometry contracts to zero size at finite $r > b$. In this case the D7-brane "ends" before reaching the horizon at $r = b$. However, for sufficiently small m/b, the D7-brane ends at the horizon, at which point the thermal S^1 contracts to zero size. Both these types of solutions are plotted in figure 1. There is a first order phase transition at the critical value of $m/b \approx 0.92$ where the two types of solution meet [20, 35]. The $\langle \bar{\psi}\psi \rangle$ condensate is non-zero on both sides of this transition, although there is a discontinuous jump in its value.
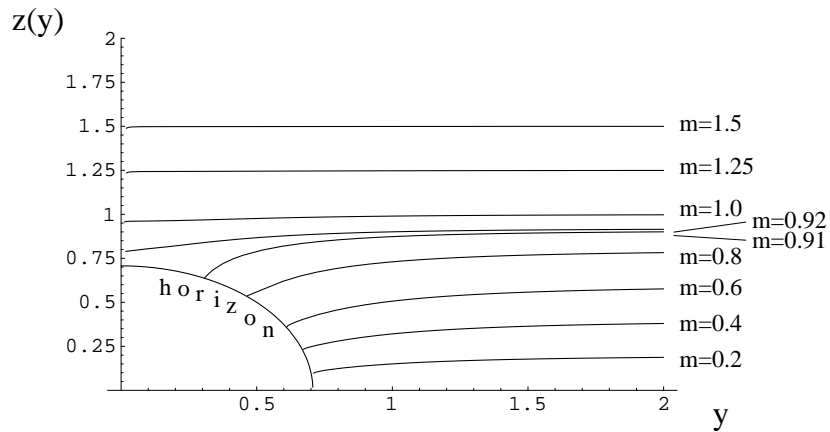

Figure 1: Brane embeddings in the AdS Schwarzschild background for different values of the quark mass. In the plot we have set $b = 1$.

Using the machinery discussed above, we can study the behaviour of the scalar VEV parametrizing the Higgs branch on either side of the transition. We evaluate the potential (27,34) numerically, with a single instanton centred at the origin for different values of the ratio m/b. The results are plotted in figure 2.
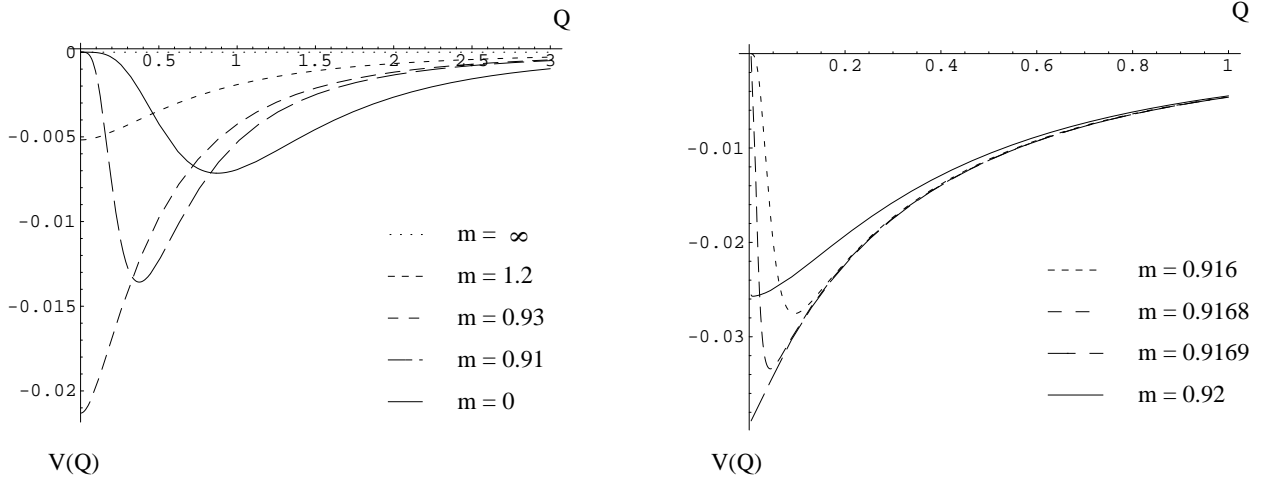

Figure 2: Potential $V(Q)$ as a function of the instanton size / Higgs VEV Q for various values of the quark mass m (we set $b = 1$ here); in fig. a) we sample the whole spectrum from $m = 0$ to $m \to \infty$, in fig. b) we vary m close to the phase transition region. The potential for $m \to \infty$ is flat and coincides with the horizontal axis.

Figure 2 shows that there are two phases for the scalar Higgs VEV matching the two phases (topologies) of the D7-brane embedding. These phases are divided by the first order phase transition at a critical value of m/b, where the parameter b of the AdS/Schwarzchild background is proportional to the temperature. In figure 2 we have chosen units such that $b = 1$. For $m/b \to \infty$, the D7 branes probe the AdS-like part of the geometry, such that the potential is flat (dotted line on the left hand side of figure 2). For all other values of m/b, the potential approaches this constant asymptotic value from below for large Q. When the value of m/b is decreased, starting at $m/b \to \infty$, a minimum of the potential forms at $Q = 0$, with increasing depth. At the critical value $m_c/b \simeq 0.92$, a new solution for the brane embedding appears, for which the potential has a minimum at $Q \neq 0$. For $m/b \to 0$, V has a maximum at $Q = 0$ and a minimum of decreasing depth for $Q \neq 0$. At $m/b = 0$ (solid line in the plot) there is still a shallow minimum of V at $Q \simeq 0.85b$. Thus, the Higgs VEV is an order parameter for the first order phase transition. We emphasize that the symmetry breaking phase occurs at small m/b, or high temperature, contrary to the usual expectation. We plot the value of the Higgs VEV as a function of m/b in Figure 3.
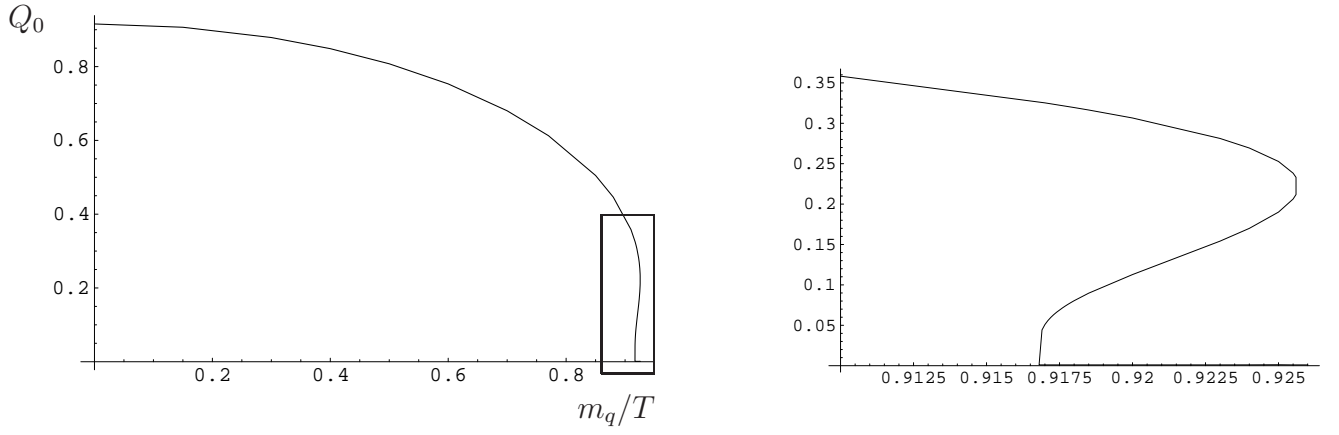
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Figure 3: Position of the minimum of the potential Q_0 versus the bare quark mass m_q, zoom of the critical region.

Note that the expectation value Q_0 is a multivalued function of m/b in a region near the first order critical point. There are several regular embeddings in this region, with the true vacuum corresponding to the solution with lower free energy.

Thus we find that for $T \geq T_c$, the theory is driven to a Higgs phase or, more precisely, a point on the mixed Coulomb-Higgs branch. We have not determined whether or not this vacuum is merely metastable, which would be the case if the origin of the Coulomb branch has lower free energy.

Since the less supersymmetric phase occurs for $T \geq T_c$, the phase transition discussed here is quite different from the deconfinement transition. Nevertheless it is interesting to note that our phase transition is first order, as is found in lattice simulations of the deconfinement transition in large N $SU(N)$ gauge theories [52].

4.4   Geometric aspects of the transition

It is worth stressing the link between the transition of the Higgs VEV between the two phases and the change in the embedding topology of the D7 brane. From figures 1 and 2, we see that the Higgs potential only has a non-zero minimum when the D7-branes reach the horizon. There is a repulsive effect in the vicinity of the horizon which causes the instanton to expand. In figure 4 we plot the value of the Higgs VEV versus the minimum value of the coordinate y to which the D7 branes extend. When the D7-branes do not reach the horizon, the S^3 component of their geometry contracts to zero size at the endpoint, which occurs at $y = 0$. On the other hand, when the D7-branes reach the horizon, the S^3 component of their geometry remains finite size and $y \neq 0$ at the endpoint. The dependence of the VEV on the minimum value of y is almost a straight line with slope roughly equal to 1.25. The solutions that reach the horizon are

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distributed along this line, while the ones which end before reaching the horizon accumulate at the origin.

The phase transition region ($m/b \simeq 0.91$) has some additional interesting structure. In this region there are three regular solutions for the D7 embedding for each value of m_q (see [35, 22]). Some of them end on the horizon and some do not. The three flows are shown for a particular case in fig. 5a. These different embeddings give different predictions for the Higgs VEV. We also show the potential for Q on each of these embeddings in figure 5b and the minimum vs m can be found in figure 3b. There are three values of Q supplied in this transition region. Clearly there will be a first order transition where the embedding jumps from solutions ending on and off the horizon as m is changed, corresponding to a discontinuous transition in the value of the scalar VEV.

5   Conclusions

We have investigated the thermodynamics of a large N_c strongly coupled $\mathcal{N} = 2$ gauge theory, using AdS/CFT duality to compute the potential which is generated on the Higgs branch at finite temperature and isospin chemical potential. In the AdS description, this potential is a potential on the moduli space of instantons.

We have shown non-perturbatively that the chemical potential destabilizes the Higgs potential, giving rise to Bose-Einstein condensation. The $\mathcal{O}(\mu^2)$ term in the potential can be computed exactly, as a consequence of the non-renormalization of the metric on the Higgs branch, which is correctly reproduced by instanton dynamics in the AdS description. The instability due to the chemical potential is not cured at finite temperature.

It would be very interesting to introduce a chemical potential for Baryon number in the AdS setting. Unlike the chemical potential for isospin, this is known to be very difficult to study on the lattice due to a complex fermionic determinant. In the AdS setting, it is also easier to introduce a chemical potential for the isospin, as we have done here, although the problems arising for baryon number are probably much more tractable than those which arise on the lattice.
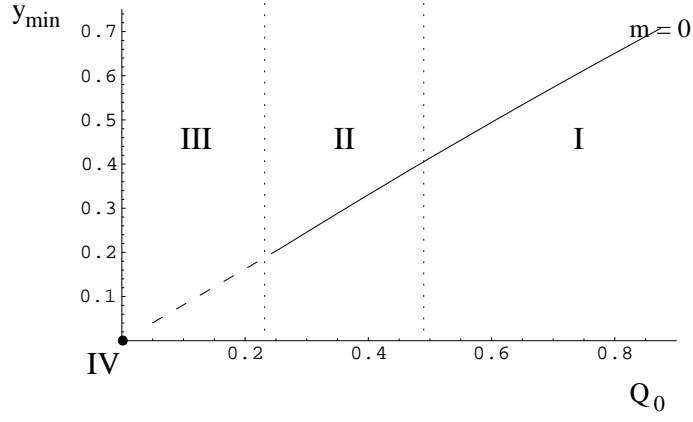
Figure 4: Position of the endpoint y_{min} of the embedded brane on the horizon versus the potential minimum Q_0. Region I corresponds to solutions with $0 \leq m_q < 0.91$. In this range, for each m_q there is only one regular D7 embedding, reaching the horizon. In the phase transition region $(0.913 \lesssim m_q \lesssim 0.926)$, there are three embeddings for each m_q (see the text for details). Of the two that reach the horizon one lies on the straight line in region II, the other in region III (dashed line).[8] The solution that does not touch the horizon has $y_{min} = Q_0 = 0$, thus lying at the origin IV (we use a dot to describe this kind of solutions). Finally for large enough quark masses $(m_q > 0.93)$, there is again only one regular embedding for each m_q, which does not reach the horizon. These solutions all accumulate in the origin IV.
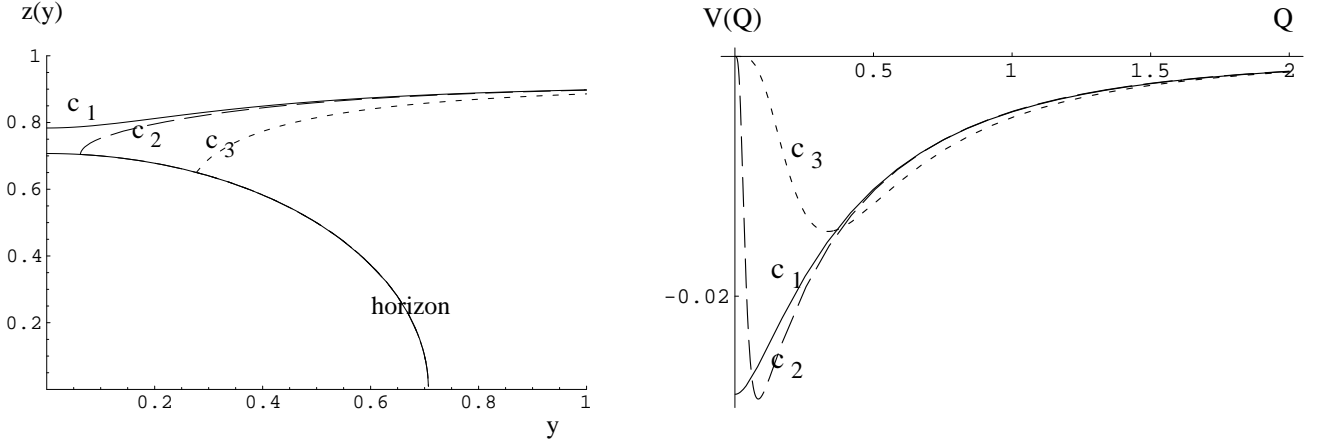

Figure 5: Solutions in critical region: There are three condensate values which give different regular solutions for the each quark mass. $m_q = 0.918$ in the plot with condensate values $c_1 = -0.0165$, $c_2 = -0.0172$, $c_3 = -0.0265$.

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[8]There are two small gaps in the plot, one between region II and III and one between III and IV. It is worth stressing that quark masses for solutions in these areas all lie in two very narrow ranges around $m_q \simeq 0.925$ and $m_q \simeq 0.916$. At a certain point the fine-tuning required to find the solutions which should fill the gaps exceeds the numerical accuracy used to solve the equation of motion.

It was known previously that this theory has a first order phase transition at a critical temperature of order of the quark mass, and that the chiral quark condensate $\langle \bar{\psi}\psi \rangle$ varies discontinuously across the transition [20, 35, 22]. By computing the effective potential on a mixed Coulomb-Higgs branch, we have found that the Higgs VEV is an order parameter for this transition. The VEV is zero for $0 < T \leq T_c$ and non-zero for $T > T_c$. For vanishing quark mass, in which case a pure Higgs branch with vanishing Coulomb-branch moduli exists, the effective potential suggests that the finite temperature theory is in a (stable) Higgs phase. Although there are numerous examples of high temperature symmetry breaking in non-supersymmetric theories, our strong coupling result differs from the typical weak coupling behavior of renormalizable supersymmetric theories with flat directions, for which the origin of moduli space becomes a stable or metastable vacuum at finite temperature.

Another application of the machinery we have developed here would be to compute the effective potential generated on the Higgs branch by other supersymmetry breaking deformations of the $\mathcal{N} = 2$ theory, besides the thermal deformations we discussed here. For example, there has been considerable interest in deformations that give rise to electrically confining theories. In principle, for all such theories one should check that the scalar potential has a minimum at the origin of the Higgs branch, before computing physical observables such as meson spectra. Moreover, in the future it might be possible with these techniques to reproduce the Affleck-Dine-Seiberg runaway scalar potential of $\mathcal{N} = 1$ supersymmetric QCD [54, 55], which arises when the number of flavors is less than the number of colors.

Acknowledgements

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Strong coupling effective Higgs potential and a first order thermal phase transition from AdS/CFT duality

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Abstract

We use AdS/CFT duality to study the thermodynamics of a strongly coupled $\mathcal{N} = 2$ supersymmetric large N_c $SU(N_c)$ gauge theory with $N_f = 2$ fundamental hypermultiplets. At finite temperature T and isospin chemical potential μ, a potential on the Higgs branch is generated, corresponding to a potential on the moduli space of instantons in the AdS description. For $\mu = 0$, there is a known first order phase transition around a critical temperature T_c. We find that the Higgs VEV is a suitable order parameter for this transition; for $T > T_c$, the theory is driven to a non-trivial point on the Higgs branch. For $\mu \neq 0$ and $T = 0$, the Higgs potential is unbounded from below, leading to an instability of the field theory due to Bose-Einstein condensation.

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1   Introduction

In its original form, AdS/CFT duality [1, 2, 3] relates theories of closed strings in asymptotically AdS spaces to large N_c gauge theories with matter in the adjoint representation. Fields in the fundamental representation may be added by including an open string sector through the introduction of branes probing the supergravity background. Much effort has gone into studying dualities of this type, motivated largely by the goal of finding a supergravity background dual to QCD.

The first example of AdS/CFT duality for a theory with fundamental representations related a conformal $\mathcal{N} = 2$ $Sp(N)$ gauge theory to string theory in $AdS_5 \times S^5/Z_2$, with D7-branes wrapping the Z_2 fixed surface with geometry $AdS_5 \times S^3$ [4, 5]. A different approach to flavour in AdS/CFT has been considered in [6]. In [7], the duality of [4, 5] was extended to an $\mathcal{N} = 2$ $SU(N_c)$ theory with N_f massive fundamental hypermultiplets, essentially by removing the $\mathbb{Z}_2$ orientifold, which was justified by the fact that a probe D7-brane wrapping a contractible S^3 does not lead to a tadpole requiring cancellation. Although the field theory is not asymptotically free, it has a UV fixed point in the strict $N_c \to \infty$ limit. Following this, there have been a number of papers generalizing the duality to confining theories with fundamental representations [8] -[19], including non-supersymmetric examples in which spontaneous chiral symmetry breaking by a $\bar{\psi}\psi$ quark condensate occurs [20] -[29].

At zero temperature and vanishing quark mass, the $\mathcal{N} = 2$ gauge theory has a non-trivial Higgs branch, i.e. a moduli space of vacua on which the scalar components of the fundamental hypermultiplets have expectation values. For finite quark mass, the moduli space includes a mixed Coulomb-Higgs branch: the fundamental hypermultiplets may have non-zero expectation values if the vector multiplet scalars have particular VEV's equal to the quark mass. For both the massless and massive case, the moduli space of field theory is described in the AdS picture by instanton configurations on the D7-branes [30, 31, 32, 33]. Self dual field strengths are solutions of the D7-brane equations of motion due to a conspiracy between the Yang-Mills and Wess-Zumino terms in the D7-brane action. In general, this conspiracy is destroyed by supersymmetry breaking deformations, which give rise to a potential on the moduli space of instantons, corresponding to an effective potential on the Higgs branch. We will compute this potential at finite isospin chemical potential and temperature. We focus on the potential generated on a slice of the moduli space corresponding to a single instanton centered at the origin. This slice is parameterized by the instanton size, which is dual to a particular Higgs VEV.

At finite isospin chemical potential, we find the expected result that the theory is destabilized via an effective negative mass squared for the moduli, leading to Bose-Einstein condensation. The negative mass squared term in the effective potential is related to the metric on the Higgs

branch, which is correctly reproduced by the dynamics of spinning instantons on the D7-branes. To stabilize the theory, a positive scalar mass which is larger than the chemical potential would need to be introduced.

Next we consider finite temperature deformations. Some finite temperature properties of theories with fundamental representations have been studied previously using AdS/CFT duality [20, 22, 34]. As shown in [20, 35], there is a finite temperature first order phase transition as the ratio of the temperature to the quark mass is varied in the $\mathcal{N} = 2$ theory of [7] (a similar transition was discussed in [22]). In the AdS description, this transition corresponds to a change in the topology of the D7-brane embedding in an AdS-Schwarzschild background.

We explore the finite temperature behavior of this theory in more detail by computing the effective potential generated on the Higgs component of the moduli space. We find that the Higgs expectation value is an order parameter for the first order phase transition. At $0 < T \leq T_c$ and $\mu = 0$, the Higgs VEV is driven to the origin of moduli space. However for $T > T_c$ we find a surprise; the instanton size is driven towards a non-zero value, suggesting the existence of a vacuum in which the theory is higgsed. This is in contrast to the weak coupling behavior of the theory, for which the one-loop finite temperature effective potential implies that the origin of moduli space is at least metastable.

The organisation of this paper is as follows. In section 2, we review the AdS description [30, 31] of the mixed Coulomb-Higgs branch in the Karch-Katz $\mathcal{N} = 2$ theory [7]. In section 3, we consider the same theory at finite chemical potential. Section 4 is devolved to the finite temperature case. In section 5 we briefly summarize our results and discuss future developments.

2   SUGRA dual of an $\mathcal{N} = 2$ theory with fundamental representations

We consider an $\mathcal{N} = 2$ gauge theory which is dual [7] to string theory in $AdS_5 \times S^5$ with N_f D7-branes wrapping a surface which is asymptotically $AdS_5 \times S^3$. The matter content of this gauge theory is that of the $\mathcal{N} = 4$ $SU(N_c)$ gauge theory together with N_f massive hypermultiplets in the fundamental representation. In $\mathcal{N} = 1$ superspace, the Lagrangian is

$$
\begin{aligned}
\mathcal{L} =& \mathrm{Im} \left[\tau \int d^2\theta d^2\bar{\theta} \left(\mathrm{tr}\,(\bar{\Phi}_I e^V \Phi_I e^{-V}) + Q_i^\dagger e^V Q^i + \tilde{Q}_i^\dagger e^{-V} \tilde{Q}^i \right) \right. \\
&+ \left. \tau \int d^2\theta \left(\mathrm{tr}\,(\mathcal{W}^\alpha \mathcal{W}_\alpha) + W \right) + \tau \int d^2\bar{\theta} \left(\mathrm{tr}\,(\overline{\mathcal{W}}_{\dot{\alpha}} \overline{\mathcal{W}}^{\dot{\alpha}}) + \overline{W} \right) \right],
\end{aligned}
\tag{1}
$$

where the superpotential W is

$$
W = \mathrm{tr}(\epsilon_{IJK} \Phi_I \Phi_J \Phi_K) + \tilde{Q}_i (m + \Phi_3) Q^i.
\tag{2}
$$

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Coordinates									
0	1	2	3	4	5	6	7	8	9
D3									
D7									
x^μ				y^m (or y, Ω_3)				Z^i	
				r					

Table 1: Index conventions

The superfields Q^i and $\tilde{Q}_i$, labeled by the flavor index $i = 1 \cdots N_f$, comprise the $\mathcal{N} = 2$ fundamental hypermultiplets.

This theory is not asymptotically free and, at finite N_c, the corresponding string background suffers from an uncancelled tadpole. However, as in [7], we focus strictly on the $N_c \to \infty$ limit with fixed N_f. In this case there is a non-trivial UV fixed point for the 't Hooft coupling, while the dual AdS string background does not suffer from a tadpole problem since the probe D7-branes wrap a contractible S^3.

In coordinates which will be convenient for our purposes, the $AdS_5 \times S^5$ background is

$$ds^2 = \frac{r^2}{R^2}dx^\mu dx^\mu + \frac{R^2}{r^2}(dy^2 + y^2 d\Omega_3^2 + \sum_{i=1}^{2} dZ^i dZ^i), \tag{3}$$

$$e^\Phi = g_s,$$

$$F_{(5)} = dC_{(4)} = 4R^4(V_{S^5} + {}^* V_{S^5}), \qquad C_{(4)}|_{0123} = \frac{r^4}{R^4} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,$$

$$R^4 = 4\pi g_s N_c \alpha'^2,$$

where V_{S^5} is the volume form on S^5, and $r^2 \equiv y^2 + Z^i Z^i$. The 't Hooft coupling in the dual gauge theory is $\lambda = g^2 N_c = g_s N_c$.

The fundamental hypermultiplets arise from N_f D7-branes embedded in this geometry. Their action is

$$S_{D7} = -T_7 \int d^4x d^4y \sqrt{-\det G_{ab}|_{PB}} = -T_7 \int d^4x \, dy \, d\Omega_3 \, y^3 \sqrt{1 + (Z_i')^2}, \tag{4}$$

where G_{PB} is the pull-back of the metric, a prime indicates a derivative with respect to y, and we have assumed an embedding independent of the coordinates x as well as the coordinates on S^3. There is a manifest minimum of the action when $Z_i' = 0$. We will choose

$$Z^1 = m, \quad Z^2 = 0, \tag{5}$$

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other solutions being related by the $U(1)_R$ symmetry corresponding to rotations in the plane spanned by the Z^i. The induced metric is

$$ds^2 = \frac{r^2}{R^2} dx_{//}^2 + \frac{R^2}{r^2}(dy^2 + y^2 d\Omega_3^2), \tag{6}$$

with $r^2 = y^2 + m^2$. The parameter m corresponds to the mass of the fundamental hypermultiplets. For $m = 0$, the geometry is $AdS_5 \times S^3$, while for $m \neq 0$, the geometry approaches $AdS^5 \times S^3$ at large r. The S^3 component of the D7-geometry contracts to zero size at $r = m$. So long as N_f is held fixed in the limit $N_c \to \infty$ with fixed $\lambda = g_s N_c \gg 1$, one can neglect the back-reaction of the D7-branes on the bulk geometry.

2.1 The Higgs branch

When the theory has massless quarks, the fundamental scalars q^i and $\tilde{q}_i$ (denoting bottom components of chiral superfields by lowercase letters,) have non-zero expectation values on the Higgs branch[1] while the adjoint scalar ϕ_3 of the $\mathcal{N} = 2$ vector multiplet vanishes. For non-zero mass, m, and vanishing ϕ_3, the fundamental hypermultiplets are massive and a pure Higgs branch does not exist. However there is a mixed Coulomb-Higgs branch when ϕ_3 has an expectation value such that some components of the hypermultiplets are massless. An example of a point on a mixed Coulomb-Higgs branch is given by a diagonal ϕ_3 for which all but the last k entries are vanishing,

$$\phi_3 = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & -m & & \\ & & & & \ddots & \\ & & & & & -m \end{pmatrix}. \tag{7}$$

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[1]The scalars ϕ_1 and ϕ_2 belonging to the adjoint hypermultiplet may also have non-zero expectation values on the Higgs branch.

In this case, the F-flatness equations $\tilde{q}_i(\phi_3 + m) = (\phi_3 + m)q^i = 0$ permit fundamental hyper-multiplet expectation values in which only the last k entries of q^i and $\tilde{q}_i$ are non-zero,

$$q^i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_1^i \\ \vdots \\ \alpha_k^i \end{pmatrix}, \qquad \tilde{q}_i = \begin{pmatrix} 0 & \cdots & 0 & \beta_{1i} & \cdots & \beta_{ki} \end{pmatrix}. \qquad (8)$$

There are additional F and D-flatness constraints which we have not explicitly written.

In string theory, nonzero entries in (8) physically correspond to D3-branes which are coincident with and dissolved within the D7-branes. Dissolved D3-branes can be viewed as instantons in the eight-dimensional world-volume theory on the D7-branes [36], due to the Wess-Zumino coupling

$$S_{WZ} = \frac{T_7}{g_s}(2\pi\alpha')^2 \int C|_{PB}^{(4)} \wedge \text{tr}(F \wedge F). \qquad (9)$$

There is a one to one map between the moduli space of Yang-Mills instantons and the Higgs branch of this $\mathcal{N} = 2$ theory. The ADHM constraints from which instantons are constructed [37] are equivalent to the F and D-flatness equations [38, 39] (see also [40] for a review).

2.2   Supergravity description of the Higgs branch

Because of the known one-one correspondence between instantons and the Higgs branch, one expects that instantons solve the equations of motion of the non-Abelian[2] Dirac-Born-Infeld action describing D7-branes embedded in (3) according to (5). The existence such solutions is a non-trivial consequence of AdS/CFT duality [30, 31, 41].

The effective action describing D7-branes in the AdS background (3) is

$$S = -T_7(2\pi\alpha')^2 \left(-\frac{1}{g_s} \int d^8\xi \, C^{(4)} \wedge \text{tr}(F \wedge F) + \int d^8\xi \, \frac{e^{-\Phi}}{4} \sqrt{-\det G} \, \text{Tr}\left(F_{\alpha\beta}F^{\alpha\beta} \right) \right) + \cdots, \qquad (10)$$

where we have not written terms involving fermions and scalars. This action is the sum of a Wess-Zumino term, a Yang-Mills term, and an infinite number of corrections at higher orders in α' indicated by $\cdots$ in (10). The correspondence between instantons and the Higgs branch suggests that the equations of motion should be solved by field strengths which are self-dual with respect to a flat four-dimensional metric.

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[2]The existence of a Higgs branch requires at least two flavors, or two D7-branes.

In this paper, we work to leading order only in the large 't Hooft coupling expansion generated by AdS/CFT duality, which allows us to neglect the higher order terms in the α' expansion[3] of the action. Constraints on unknown higher order terms arising from the existence of instanton solutions, as well as the exactly known metric on the Higgs branch, were discussed in [30, 41].

The induced metric (6) can be written as

$$ds^2 = \frac{r^2}{R^2}dx^\mu dx^\mu + \frac{R^2}{r^2}\sum_{m=1}^{4} dy^m dy^m , \tag{11}$$

with $r^2 = y^m y^m + m^2$. Field strengths which are self dual with respect to the flat four-dimensional metric $ds^2 = \sum_{m=1}^{4} dy^m dy^m$ solve the equations of motion, due to a conspiracy between the Wess-Zumino and Yang-Mills term. Inserting the explicit AdS background values (3) for the metric and Ramond-Ramond four-form, into the action (10) for D7-branes embedded according to (5), with non-trivial field strengths only in the y^m directions, gives

$$\begin{aligned} S &= -T_7(2\pi\alpha')^2 \int d^4x \, d^4y \, \frac{r^4}{4g_s R^4} \left(-\frac{1}{2}\epsilon_{mnrs}F_{mn}F_{rs} + F_{mn}F_{mn} \right) \\ &= -T_7(4\pi\alpha')^2 \int d^4x \, d^4y \, \frac{r^4}{4g_s R^4} F_-^2 , \end{aligned} \tag{12}$$

where $F_{mn}^- = \frac{1}{2}(F_{mn} - \frac{1}{2}\epsilon_{mnrs}F_{rs})$. Field strengths $F_{mn}^- = 0$, which are self-dual with respect to the flat metric $dy^m dy^m$, manifestly solve the equations of motion[4]. These solutions correspond to points on the Higgs branch of the dual $\mathcal{N} = 2$ theory. If $m \neq 0$ this is, strictly speaking, a point on the mixed Coulomb-Higgs branch, with expectation values of the form (7), (8). We emphasize that in order to neglect the back-reaction due to dissolved D3-branes, we are considering a portion of the moduli space for which the instanton number k is fixed in the large N_c limit.

2.3 A slice of the Higgs branch

For simplicity, we consider the case $N_f = 2$, which is the minimum value for which a non-trivial Higgs branch exists. We will focus on a slice of the Higgs branch (or mixed Coulomb-Higgs branch) corresponding to a single instanton centered at the origin, $y^m = 0$.

In "singular gauge", the $SU(2)$ instanton (using the same conventions as [40]) is given by

$$A_\mu = 0, \qquad A_m = \frac{2Q^2\bar\sigma_{nm}y_n}{y^2(y^2 + Q^2)}, \tag{13}$$

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[3]The dimensionless expansion parameter is $\alpha'/R^2 = 1/\sqrt{\lambda}$.
[4]Anti-instantons, with $F^+ = 0$, correspond to non-supersymmetric configurations which do not solve the equations of motion.

where Q is the instanton size, and

$$\bar{\sigma}_{mn} \equiv \frac{1}{4}(\bar{\sigma}_m \sigma_n - \bar{\sigma}_n \sigma_m), \qquad \sigma_{mn} \equiv \frac{1}{4}(\sigma_m \bar{\sigma}_n - \sigma_n \bar{\sigma}_m),$$
$$\sigma_m \equiv (i\vec{\tau}, 1_{2\times 2}), \qquad \bar{\sigma}_m \equiv \sigma_m^\dagger = (-i\vec{\tau}, 1_{2\times 2}). \tag{14}$$

with $\vec{\tau}$ being the three Pauli-matrices. We choose singular gauge, as opposed to the regular gauge in which $A_n = 2\sigma_{mn}y^m/(y^2 + Q^2)$, because of the improved asymptotic behavior at large y. In the AdS setting, the Higgs branch should correspond to a normalizable deformation of the AdS background at the origin of the moduli space. We see that the solution falls to zero at large y and the deformation in the interior is controlled by the dimension two parameter Q^2 which corresponds to the VEV of $\tilde{q}^i q_i$. The choice of singular gauge will be of particular use when computing the effects of the chemical potential. The singularity of (13) at $y^m = 0$ is not problematic for computations of physical (gauge invariant) quantities.

For $m = 0$ and zero instanton number, the background geometry preserves the symmetries $SO(2,4) \times SU(2)_L \times SU(2)_R \times U(1)_R \times SU(2)_f$. The $SO(2,4)$ isometry of the AdS^5 factor corresponds to the conformal symmetry of the dual gauge theory. The $SU(2)_L \times SU(2)_R \sim SO(4)$ acts in the obvious way on the coordinates y^m. The $SU(2)_L$ factor corresponds to a global symmetry, while $SU(2)_R$ and $U(1)_R$, which acts on (Z^1, Z^2), correspond to R-symmetries[5]. $SU(2)_f$ is a gauge transformation on the D7-branes which is constant at the boundary of AdS. The action at the boundary of AdS corresponds to the flavor symmetry of the dual gauge theory.

In the presence of the instanton (13), and for $m \neq 0$, the symmetries are broken to $SO(1,3) \times SU(2)_L \times \text{diag}(SU(2)_R \times SU(2)_f)$ corresponding to a point on the mixed Coulomb-Higgs branch

$$q_{i\alpha} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \epsilon_{i\alpha} Q \end{pmatrix}, \quad \phi_\alpha = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \\ & & & -m \end{pmatrix}, \tag{15}$$

where $q_{i\alpha}$ are the scalar components of the fundamental hypermultiplets, with flavor index $i = 1, 2$ and $SU(2)_R$ index $\alpha = 1, 2$. The scalars ϕ_α belong to the adjoint hypermultiplet, while ϕ_3 is the adjoint scalar belonging to the vector-multiplet.

In general, one expects that supersymmetry breaking deformations, such as a finite chemical potential or temperature, lift the vacuum degeneracy of the Higgs and Coulomb branch. In the dual supergravity description, the lifting of the Higgs branch at finite temperature occurs because the conspiracy between the Wess-Zumino and Yang-Mills terms, necessary for self-dual field strengths to be solutions, no longer occurs. The potential which is generated on the Higgs branch can be computed by evaluating the D7-brane action on the space of self-dual field

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[5]There is no apparent distinction between $SU(2)_L$ and $SU(2)_R$ in the geometry of the AdS background. The distinction arises from the gauge field couplings to the RR four-form.

strengths. In the following we will compute the potential $V(Q)$ generated on the slice of the Higgs branch dual to the single instanton (13).

3 Finite chemical potential, zero temperature

As the simplest example of a potential generated on the Higgs branch, we first consider the case of finite chemical potential and zero temperature. For two flavors, there is a U(2) flavor symmetry, with Cartan generators 1 and σ_3, corresponding to baryon number and isospin respectively. We will consider a nonzero chemical potential for the isospin[6]. To include the chemical potential [42] we allow a spurious gauge field associated with the τ^3 component of isospin to acquire a VEV, μ, in its A^0 component. This includes generic fermion and scalar Lagrangian terms for fields with isospin charge e of the form

$$\delta\mathcal{L} = -\mu e \bar{\psi} \tau^3 \gamma^0 \psi + \mu^2 e^2 |\phi|^2 \,. \tag{16}$$

The first term is a source for the fermionic isospin number density. In the path integral, this term places the theory at finite density. The second term is an unbounded scalar potential which renders the theory unstable, such that Bose-Einstein condensation is expected. We will reproduce this run-away behaviour of the scalar potential using the AdS/CFT description. Equivalently, we may set $A_\tau = 0$ by a gauge transformation $e^{i\mu e\tau}$, and perform the path integral with boundary conditions corresponding to a spinning configuration space.

AdS/CFT duality relates global symmetries of the boundary theory to gauge symmetries in the bulk supergravity. The prescription for turning on a chemical potential in the AdS description is to turn on a background (non-normalizable) flat gauge connection $\mathcal{A}_\tau = \mu$ for the associated gauge symmetry. Equivalently, one may consider a spinning AdS background. Examples of the AdS description of a chemical potential in various contexts have appeared in [43]-[48].

The effect of the chemical potential on the Higgs branch can be studied in the AdS description by computing the action for rigidly rotating instantons with moduli $\mathcal{M}^i$,

$$A_n = e^{i\mu t\sigma_3} \mathcal{A}_n^{\text{instanton}}(y^m, \mathcal{M}^i) e^{-i\mu t\sigma_3} \,. \tag{17}$$

Equivalently, we may add a background A^0 in the fixed instanton background,

$$A^0 = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \qquad A_n = A_n^{\text{instanton}} \,. \tag{18}$$

On the slice of the Higgs branch corresponding to the single instanton configurations (13) with modulus Q, the effective potential at quadratic order in μ can be determined by inserting (18)

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[6]Turning on a chemical potential for the baryon number has no apparent effect here since none of the fields on the D7 brane world volume are charged under this symmetry. This was pointed out to us by A. Karch.

into the D7-brane action $\int d^4x\, V(Q) = -S_{D7}$ giving

$$V(Q) = T_7 \frac{(2\pi\alpha')^2}{g_s} \int d^4y\, \mathrm{tr}\left(\frac{1}{2}\frac{(y^2+m^2)^2}{R^4} F^-_{mn}F^-_{mn} + 2F_{m\mu}F_{m\nu}\eta^{\mu\nu} \right. \tag{19}$$

$$\left. + \frac{R^4}{(y^2+m^2)^2} F_{\mu\nu}F_{\alpha\beta}\eta^{\mu\alpha}\eta^{\nu\beta} \right),$$

with $y^2 = y^m y^m$. We have split the action into the pieces involving F in the x and y directions, indicated by Greek and Roman indices respectively, as well as mixed terms. For the background (18), the only non-zero contribution to the potential comes from the mixed term $\mathrm{tr}\, F_{\mu m}F_{\nu m}\eta^{\mu\nu} = -\mathrm{tr}\,[A_0, A_n]^2$, giving

$$V(Q) = -T_7 \frac{2(4\pi\alpha')^2}{g_s}\mu^2 \int d^4y \frac{Q^4}{y^2(y^2+Q^2)^2} = -T_7 \frac{2(4\pi^2\alpha')^2}{g_s}\mu^2 Q^2. \tag{20}$$

Note that the term $\sqrt{g}F_{\mu m}F^{\mu m}$ in the D7-action is also the term which determines the metric (two-derivative effective action) on the Higgs branch. To quadratic order in the chemical potential, the result (20) is exact. Higher dimension operators in the DBI action with two Greek indices, which have been omitted from (19), must vanish in the instanton background, otherwise there would be corrections to the Higgs branch metric in a strong 't Hooft coupling expansion [30, 31]. By the same token, the term in the potential on the Higgs branch which is quadratic in the chemical potential is given exactly by the $F_{\mu m}F^{\mu m}$ term.

In light of (20), the $\mathcal{N} = 2$ theory is unstable at finite chemical potential. Physically, the instability is due to Bose-Einstein condensation. This instability will not be cured by taking into account the higher derivative terms in the effective action, whose contribution does not effect the large Q asymptotics. Potentially, the instability could be removed by the inclusion of an appropriately large mass term for the scalars, either as an explicit additional perturbation, or generated radiatively from the inclusion of some other relevant perturbation. The finite temperature configurations studied below, however, will not be able to cure this instability since - as shown below - the relevant deformation, T^4, has dimension four and can at best generate a potential T^8/Q^4. This does not stabilize the theory.

4   Finite temperature, zero chemical potential

Before investigating the large 't Hooft coupling behavior of the effective potential on the Higgs branch, let us review the situation to one loop in perturbation theory. In a supersymmetric theory, the one-loop effective potential is

$$V = V_0 + \frac{1}{8}T^2 \sum_{i,j} \left| \frac{\partial^2 W}{\partial\phi_i\partial\phi_j} \right|^2, \tag{21}$$

where V_0 is the zero temperature effective potential, W is the superpotential, and ϕ_i are the scalar components of chiral superfields. For the theory we are considering, this potential drives the system towards the origin of moduli space. In fact it is often said that supersymmetric theories with flat directions do not have symmetry breaking vacua at finite temperature [49] (for some exceptions, see [50, 51]). However, for large 't Hooft coupling, we will find that the theory we consider does indeed exhibit symmetry breaking at high temperature. The origin of the Higgs branch becomes a maximum of the effective potential for $T > T_c$.

The gravitational dual of $\mathcal{N} = 4$ gauge theory at large 't Hooft coupling and finite temperature is obtained by replacing the background (3) with the AdS-Schwarzschild black-hole background [53]. The latter belongs to a general class of supergravity solutions which, in a choice of coordinates convenient for our purposes, have the form

$$ds^2 = f(r)(d\vec{x}^2 + g(r)d\tau^2) + h(r)(\sum_{m=1}^{4} dy^m dy^m + \sum_{i=1}^{2} dZ^i dZ^i),$$

$$e^{-\Phi} = \phi(r),$$

$$F^{(5)} = 4R^4(V_{S^5} + {}^* V_{S^5}) = dC_{(4)}, \qquad C_{(4)}|_{0123} = s(r)\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \,,$$

$$r^2 = y^m y^m + Z^i Z^i. \tag{22}$$

For the AdS-Schwarzschild solution, we have

$$f(r) = \frac{4r^4 + b^4}{4r^2 R^2}, \quad g(r) = \left(\frac{4r^4 - b^4}{4r^4 + b^4}\right)^2, \quad h(r) = \frac{R^2}{r^2}, \quad s(r) = \frac{r^4}{R^4}\left(1 + \frac{b^8}{16r^8}\right). \tag{23}$$

The coordinates $\vec{x}$ are the spatial coordinates of the dual gauge theory and τ is the Euclidean time direction, which is compactified on a circle of radius b^{-1}, corresponding to the inverse temperature. Note that the temperature $T \sim b$ only enters to the fourth power.

The D7 embedding in this background to describe our $\mathcal{N} = 2$ gauge theory with quarks is

$$Z^2 = 0, \qquad Z^1 = z(y), \tag{24}$$

where $y^2 \equiv y^m y^m$. The induced metric $G|_{\text{pb}}$ takes the form

$$ds_{\text{D7}}^2 = f(r)\left(d\vec{x}^2 + g(r)d\tau^2\right) + h(r)\left(dy^m dy^m + \left(\frac{z'(y)}{y}\right)^2 (y^m dy^m)^2\right), \tag{25}$$

with $r^2 = y^2 + z^2(y)$. To lowest order in α', the D7-brane action is

$$-T_7 \int d^8\xi\, e^{-\Phi}\sqrt{-\det G}|_{\text{pb}} = -2\pi^2 T_7 \int d^4x\, dy\, y^3 \phi(r) f(r)^2 \sqrt{g(r)}\, h(r)^2 \sqrt{1 + z'(y)^2}, \tag{26}$$

and the specific embeddings $z(y)$ can be found by solving the Euler-Lagrange equation with boundary conditions $z(\infty) = m$, $z'(\infty) = 0$.

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At large y the geometry returns to AdS where the asymptotic embedding solution (24) takes the form $z(y) \sim m + c/y^2$. Given m, the parameter c, corresponding to a $\bar{\psi}\psi$ condensate, is determined by requiring the solution to be smooth and normalizable. These solutions were studied in detail in [20].

The potential generated on the Higgs branch is obtained by calculating a contribution to the D7-brane action involving the gauge field F_{ab} at second order in α'. Specifically, one evaluates the action on the space of field strengths which are self-dual[7] with respect to the induced metric in the directions transverse to $\tau, \vec{x}$;

$$V = \frac{T_7(2\pi\alpha')^2}{2} \left(\frac{1}{g_s} \int d^4y \, C_{0123}^{(4)} \, \epsilon_{mnrs} \, \text{Tr} \, F_{mn}F_{rs} - \frac{1}{2} \int d^4y \, e^{-\Phi} \sqrt{-\det G} \, \text{Tr} \, F^{mn}F_{mn} \right), \quad (27)$$

where F_{mn} is self-dual with respect to the metric

$$ds_\perp^2 = h(r) \left((1 + z'(y)^2)dy^2 + y^2 d\Omega_3^2 \right) . \quad (28)$$

This metric is conformally flat. With new coordinates $\tilde{y}(y)$ such that $ds^2 = \alpha(\tilde{y})(d\tilde{y}^2 + \tilde{y}^2 d\Omega_3^2)$, the instanton configurations (self-dual field strengths) take the usual form.

In the $b = 0$ (zero temperature) limit, the analysis of section (2.2) holds and the potential (27) becomes flat. At finite temperature, the conspiracy between the Wess-Zumino term and the Yang-Mills terms fails, giving a potential on the moduli space of instantons. We now compute the form of the potential for the slice of moduli space corresponding to an instanton centered at the origin.

4.1 Computation of the Higgs Effective Potential

We wish to evaluate the D7-brane action on the space of fields strengths which are self-dual with respect to the transverse part of the metric

$$ds_\perp^2 = h(r) \left(dy^m dy^m + \left(\frac{z'(y)}{y} \right)^2 (y^m dy^m)^2 \right) .$$

This metric is conformally flat. There exists a coordinate transformation

$$\tilde{y}^m = J(y)y^m \quad (29)$$

such that the induced metric has the form

$$ds_{\text{D7}}^2 = f(r) \left(d\vec{x}^2 + g(r)d\tau^2 \right) + \frac{h(r)}{J(y)^2} d\tilde{y}^m d\tilde{y}^m . \quad (30)$$

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[7]There are couplings between world-volume scalars and field strengths at higher orders in α' which could alter the embedding. However we only consider the leading term in a large 't Hooft coupling expansion for which these couplings can be neglected.

The function $J(y)$ is obtained by solving

$$\left(\frac{J'}{J}\right)^2 + \frac{2}{y}\frac{J'}{J} - \frac{1}{y^2}z'(y)^2 = 0,\tag{31}$$

subject to the boundary condition $J(y \to \infty) = 1$. In the coordinates $\tilde{y}^m$, the single instanton centered at the origin has the usual form,

$$A_m = \frac{2Q^2 \bar{\sigma}_{nm}\tilde{y}^n}{\tilde{y}^2(\tilde{y}^2 + Q^2)}.\tag{32}$$

The embedding function $z(y)$ is determined by shooting techniques as in [20]. The potential on the Higgs branch is then given by $S_{D7} = -\int d^4x V(Q)$, or

$$V(Q) = T_7 \frac{(2\pi\alpha')^2}{2}\left(\frac{1}{g_s}\int d^4y\, C^{(4)}_{0123}\, \epsilon_{mnrs}\, \mathrm{Tr}\, F_{mn}F_{rs} - \frac{1}{2}\int d^4y\, e^{-\Phi}\sqrt{-\det G}\,\mathrm{Tr}F^{mn}F_{mn}\right)$$

$$= T_7 \frac{(2\pi\alpha')^2}{4g_s}\int d^4\tilde{y}\left(s(r) - \phi(r)f(r)^2\sqrt{g(r)}\right)\frac{96\,Q^4}{(\tilde{y}^2 + Q^2)^4}\tag{33}$$

$$= T_7 \frac{(2\pi\alpha')^2}{4g_s}Vol(S_3)\int y^3 dy\left(s(r) - \phi(r)f(r)^2\sqrt{g(r)}\right)\frac{96\,Q^4\,J(y)^4}{(J(y)^2y^2 + Q^2)^4}\left(1 + y\frac{J'(y)}{J(y)}\right).$$

For the AdS-Schwarzschild background (22,23) we obtain

$$V(Q) = T_7 \frac{6(2\pi^2\alpha')^2}{g_s R^4}\int dy\, \frac{b^8 y^3}{(J(y)^2y^2 + z(y)^2)^2}\frac{Q^4 J(y)^4}{(J(y)^2y^2 + Q^2)^4}\left(1 + y\frac{J'(y)}{J(y)}\right).\tag{34}$$

4.2 Large VEV Asymptotics

Without resorting to numerics, some qualitative features of $V(Q)$ at large Q (or large instanton size) can be determined from the large y behavior of the induced metric and RR-form given in (22,23), together with the asymptotic behaviour of the embedding,

$$z(y) = m + \frac{c}{y^2} + \cdots.\tag{35}$$

At large y we can neglect the condensate and the leading terms in the potential will depend on m and on the deformation parameter of the background b. The D7-brane action for a large instanton, which has support at large y is

$$\int d^4x V(Q) = -S_{D7}$$

$$\approx T_7 \frac{6(2\pi^2\alpha')^2}{g_s R^4}\int d^4x\int dy\, y^3\frac{Q^4}{(y^2 + Q^2)^4}\left[\frac{b^8}{y^4} - \frac{2b^8 m^2}{y^6} + \cdots\right]\tag{36}$$

$$= T_7 \frac{6(2\pi^2\alpha')^2}{g_s R^4}\left[-\frac{b^8}{Q^4}\left(\frac{1}{2}\log 2 - \frac{1}{3}\right) + \frac{2m^2 b^8}{Q^6}\left(\frac{67}{48} - 2\log 2\right) + \cdots\right].$$

This asymptotic approximation is valid when $y \geq Q$ but gives IR divergent integrals. We have evaluated the integrals down to $y/Q = 1$. In the next section we evaluate the full potential including the correct, convergent, IR behaviour.

The asymptotic potential terms are simply what one would expect on dimensional grounds. The potential must vanish as $b \to 0$ and the geometry becomes AdS. Since b enters as b^4, the first Q dependent term must take the form b^8/Q^4.

4.3   Numerical computation of $V(Q)$

To compute $V(Q)$ in general requires knowledge of the embedding function $z(y)$, which has been computed by a numerical shooting technique in [20]. Imposing boundary conditions corresponding to the large y behaviour (35), and requiring smooth behaviour in the interior, such that an RG flow interpretation is possible, leads to a dependence $c(m,b)$ of the chiral quark condensate $\langle \bar{\psi}\psi \rangle$ on the quark mass and on the temperature. Depending on the ratio m/b, there are two types of solutions, which differ by the topology of the D7-branes. At large r (or y) the geometry of the D7-branes is $AdS_5 \times S^3$ and the topology of the $r \to \infty$ boundary is $S^1 \times R^3 \times S^3$. For sufficiently large m/b, the S^3 component of the D7-geometry contracts to zero size at finite $r > b$. In this case the D7-brane "ends" before reaching the horizon at $r = b$. However, for sufficiently small m/b, the D7-brane ends at the horizon, at which point the thermal S^1 contracts to zero size. Both these types of solutions are plotted in figure 1. There is a first order phase transition at the critical value of $m/b \approx 0.92$ where the two types of solution meet [20, 35]. The $\langle \bar{\psi}\psi \rangle$ condensate is non-zero on both sides of this transition, although there is a discontinuous jump in its value.
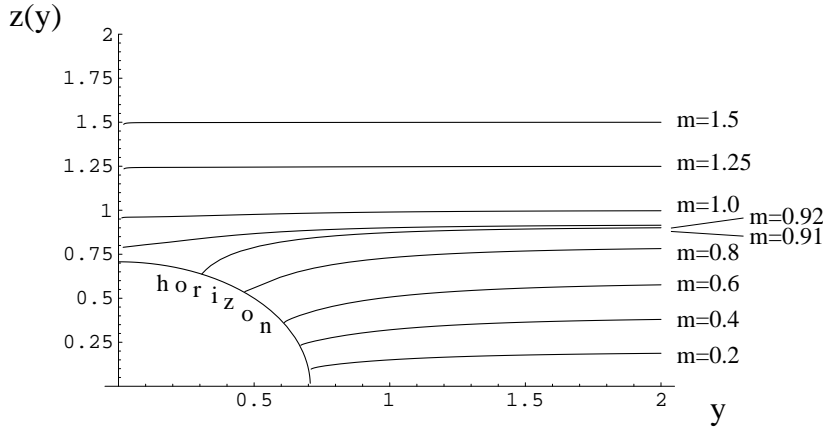

Figure 1: Brane embeddings in the AdS Schwarzschild background for different values of the quark mass. In the plot we have set $b = 1$.

Using the machinery discussed above, we can study the behaviour of the scalar VEV parametrizing the Higgs branch on either side of the transition. We evaluate the potential (27,34) numerically, with a single instanton centred at the origin for different values of the ratio m/b. The results are plotted in figure 2.
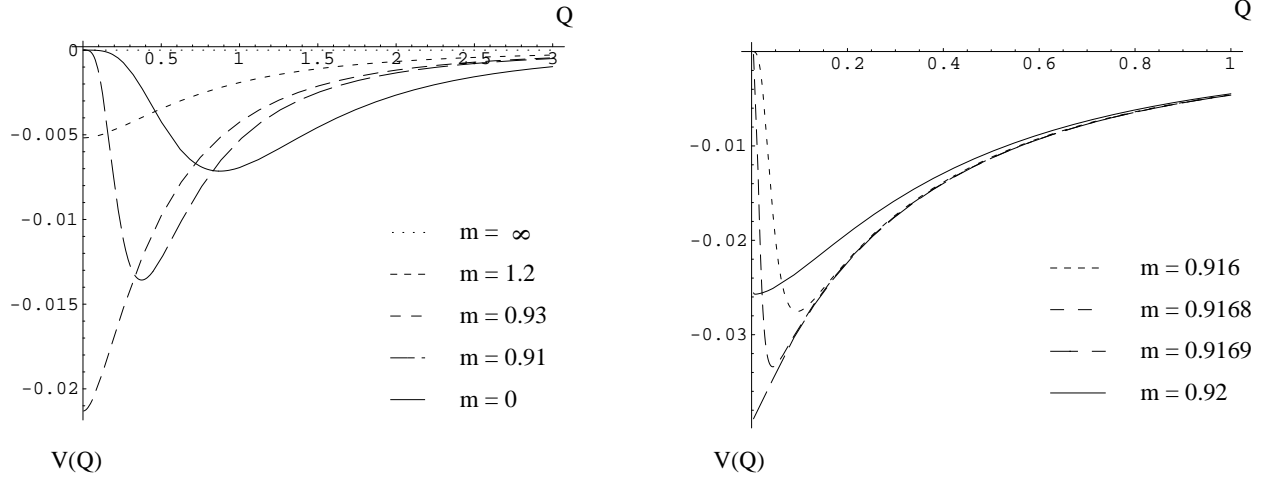

Figure 2: Potential $V(Q)$ as a function of the instanton size / Higgs VEV Q for various values of the quark mass m (we set $b = 1$ here); in fig. a) we sample the whole spectrum from $m = 0$ to $m \to \infty$, in fig. b) we vary m close to the phase transition region. The potential for $m \to \infty$ is flat and coincides with the horizontal axis.

Figure 2 shows that there are two phases for the scalar Higgs VEV matching the two phases (topologies) of the D7-brane embedding. These phases are divided by the first order phase transition at a critical value of m/b, where the parameter b of the AdS/Schwarzchild background is proportional to the temperature. In figure 2 we have chosen units such that $b = 1$. For $m/b \to \infty$, the D7 branes probe the AdS-like part of the geometry, such that the potential is flat (dotted line on the left hand side of figure 2). For all other values of m/b, the potential approaches this constant asymptotic value from below for large Q. When the value of m/b is decreased, starting at $m/b \to \infty$, a minimum of the potential forms at $Q = 0$, with increasing depth. At the critical value $m_c/b \simeq 0.92$, a new solution for the brane embedding appears, for which the potential has a minimum at $Q \neq 0$. For $m/b \to 0$, V has a maximum at $Q = 0$ and a minimum of decreasing depth for $Q \neq 0$. At $m/b = 0$ (solid line in the plot) there is still a shallow minimum of V at $Q \simeq 0.85b$. Thus, the Higgs VEV is an order parameter for the first order phase transition. We emphasize that the symmetry breaking phase occurs at small m/b, or high temperature, contrary to the usual expectation. We plot the value of the Higgs VEV as a function of m/b in Figure 3.
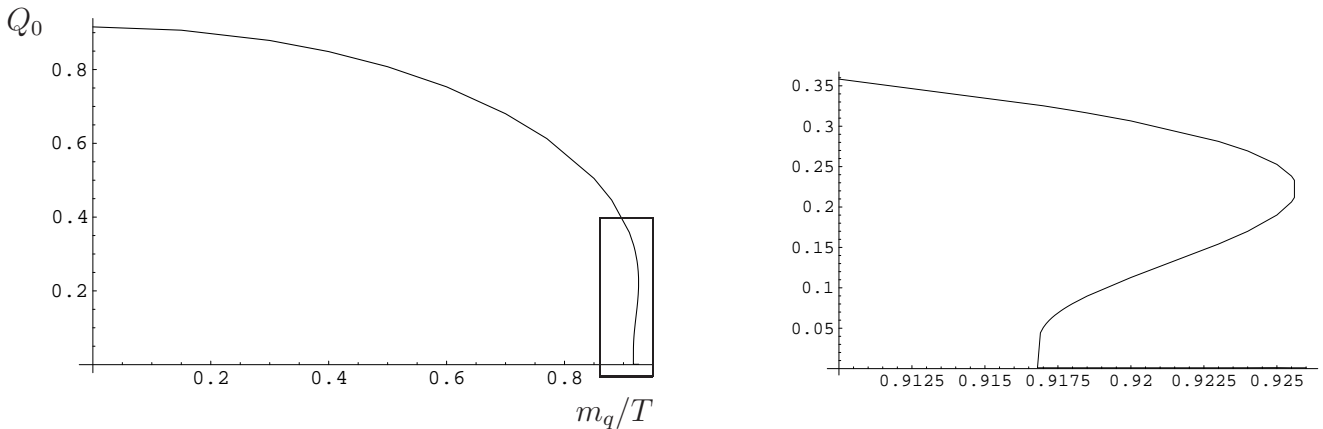
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Figure 3: Position of the minimum of the potential Q_0 versus the bare quark mass m_q, zoom of the critical region.

Note that the expectation value Q_0 is a multivalued function of m/b in a region near the first order critical point. There are several regular embeddings in this region, with the true vacuum corresponding to the solution with lower free energy.

Thus we find that for $T \geq T_c$, the theory is driven to a Higgs phase or, more precisely, a point on the mixed Coulomb-Higgs branch. We have not determined whether or not this vacuum is merely metastable, which would be the case if the origin of the Coulomb branch has lower free energy.

Since the less supersymmetric phase occurs for $T \geq T_c$, the phase transition discussed here is quite different from the deconfinement transition. Nevertheless it is interesting to note that our phase transition is first order, as is found in lattice simulations of the deconfinement transition in large N $SU(N)$ gauge theories [52].

4.4 Geometric aspects of the transition

It is worth stressing the link between the transition of the Higgs VEV between the two phases and the change in the embedding topology of the D7 brane. From figures 1 and 2, we see that the Higgs potential only has a non-zero minimum when the D7-branes reach the horizon. There is a repulsive effect in the vicinity of the horizon which causes the instanton to expand. In figure 4 we plot the value of the Higgs VEV versus the minimum value of the coordinate y to which the D7 branes extend. When the D7-branes do not reach the horizon, the S^3 component of their geometry contracts to zero size at the endpoint, which occurs at $y = 0$. On the other hand, when the D7-branes reach the horizon, the S^3 component of their geometry remains finite size and $y \neq 0$ at the endpoint. The dependence of the VEV on the minimum value of y is almost a straight line with slope roughly equal to 1.25. The solutions that reach the horizon are

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distributed along this line, while the ones which end before reaching the horizon accumulate at the origin.

The phase transition region ($m/b \simeq 0.91$) has some additional interesting structure. In this region there are three regular solutions for the D7 embedding for each value of m_q (see [35, 22]). Some of them end on the horizon and some do not. The three flows are shown for a particular case in fig. 5a. These different embeddings give different predictions for the Higgs VEV. We also show the potential for Q on each of these embeddings in figure 5b and the minimum vs m can be found in figure 3b. There are three values of Q supplied in this transition region. Clearly there will be a first order transition where the embedding jumps from solutions ending on and off the horizon as m is changed, corresponding to a discontinuous transition in the value of the scalar VEV.

5 Conclusions

We have investigated the thermodynamics of a large N_c strongly coupled $\mathcal{N} = 2$ gauge theory, using AdS/CFT duality to compute the potential which is generated on the Higgs branch at finite temperature and isospin chemical potential. In the AdS description, this potential is a potential on the moduli space of instantons.

We have shown non-perturbatively that the chemical potential destabilizes the Higgs potential, giving rise to Bose-Einstein condensation. The $\mathcal{O}(\mu^2)$ term in the potential can be computed exactly, as a consequence of the non-renormalization of the metric on the Higgs branch, which is correctly reproduced by instanton dynamics in the AdS description. The instability due to the chemical potential is not cured at finite temperature.

It would be very interesting to introduce a chemical potential for Baryon number in the AdS setting. Unlike the chemical potential for isospin, this is known to be very difficult to study on the lattice due to a complex fermionic determinant. In the AdS setting, it is also easier to introduce a chemical potential for the isospin, as we have done here, although the problems arising for baryon number are probably much more tractable than those which arise on the lattice.
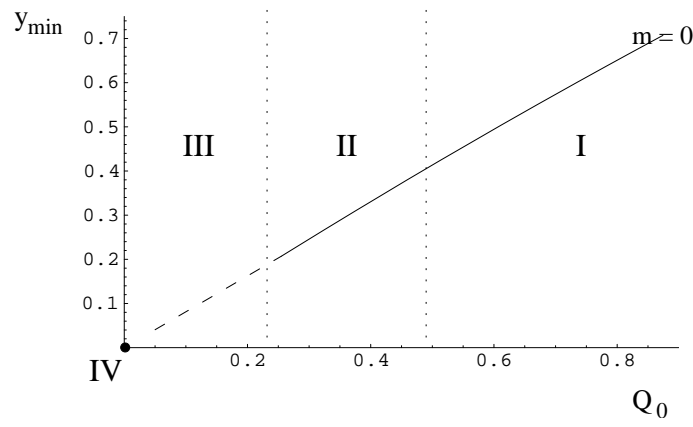
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Figure 4: Position of the endpoint y_{min} of the embedded brane on the horizon versus the potential minimum Q_0. Region I corresponds to solutions with $0 \leq m_q < 0.91$. In this range, for each m_q there is only one regular D7 embedding, reaching the horizon. In the phase transition region $(0.913 \lesssim m_q \lesssim 0.926)$, there are three embeddings for each m_q (see the text for details). Of the two that reach the horizon one lies on the straight line in region II, the other in region III (dashed line).[8] The solution that does not touch the horizon has $y_{min} = Q_0 = 0$, thus lying at the origin IV (we use a dot to describe this kind of solutions). Finally for large enough quark masses $(m_q > 0.93)$, there is again only one regular embedding for each m_q, which does not reach the horizon. These solutions all accumulate in the origin IV.
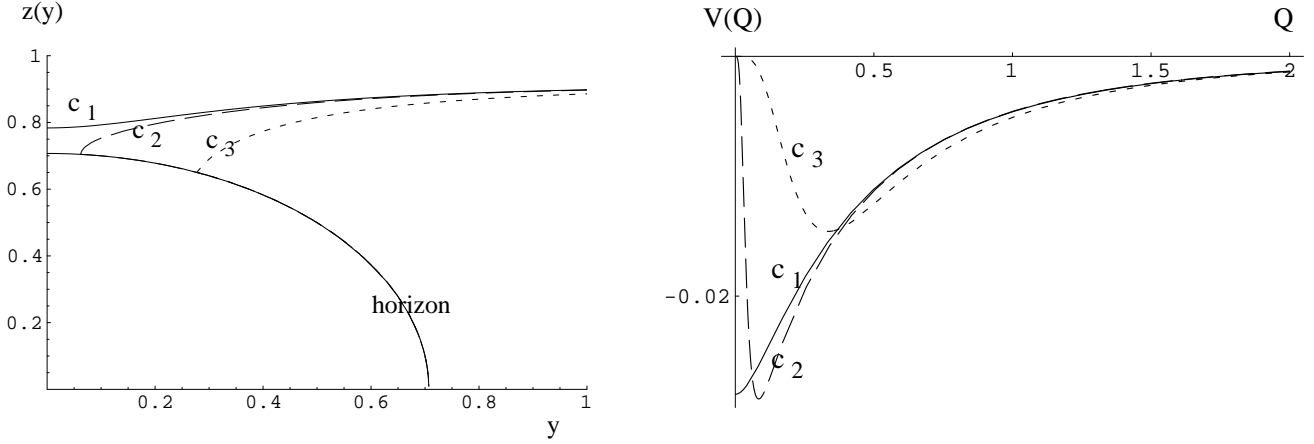

Figure 5: Solutions in critical region: There are three condensate values which give different regular solutions for the each quark mass. $m_q = 0.918$ in the plot with condensate values $c_1 = -0.0165$, $c_2 = -0.0172$, $c_3 = -0.0265$.

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[8]There are two small gaps in the plot, one between region II and III and one between III and IV. It is worth stressing that quark masses for solutions in these areas all lie in two very narrow ranges around $m_q \simeq 0.925$ and $m_q \simeq 0.916$. At a certain point the fine-tuning required to find the solutions which should fill the gaps exceeds the numerical accuracy used to solve the equation of motion.

It was known previously that this theory has a first order phase transition at a critical temperature of order of the quark mass, and that the chiral quark condensate $\langle \bar{\psi}\psi \rangle$ varies discontinuously across the transition [20, 35, 22]. By computing the effective potential on a mixed Coulomb-Higgs branch, we have found that the Higgs VEV is an order parameter for this transition. The VEV is zero for $0 < T \leq T_c$ and non-zero for $T > T_c$. For vanishing quark mass, in which case a pure Higgs branch with vanishing Coulomb-branch moduli exists, the effective potential suggests that the finite temperature theory is in a (stable) Higgs phase. Although there are numerous examples of high temperature symmetry breaking in non-supersymmetric theories, our strong coupling result differs from the typical weak coupling behavior of renormalizable supersymmetric theories with flat directions, for which the origin of moduli space becomes a stable or metastable vacuum at finite temperature.

Another application of the machinery we have developed here would be to compute the effective potential generated on the Higgs branch by other supersymmetry breaking deformations of the $\mathcal{N} = 2$ theory, besides the thermal deformations we discussed here. For example, there has been considerable interest in deformations that give rise to electrically confining theories. In principle, for all such theories one should check that the scalar potential has a minimum at the origin of the Higgs branch, before computing physical observables such as meson spectra. Moreover, in the future it might be possible with these techniques to reproduce the Affleck-Dine-Seiberg runaway scalar potential of $\mathcal{N} = 1$ supersymmetric QCD [54, 55], which arises when the number of flavors is less than the number of colors.

Acknowledgements

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